Available at WWW.MATHEMATICSWEB.ORG POWERED BY SCIENCE DIRECT

ACADEMIC PRESS

Journal of Algebra 259 (2003) 572–580

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

A Borel–Weil–Bott type theorem for group completions ☆

Syu Kato

Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba Meguro-ku, 153-8914 Tokyo, Japan

Received 4 February 2002

Communicated by Corrado De Concini

Abstract

We prove a character formula of cohomologies of line bundles on the wonderful completion of an adjoint semisimple algebraic group in the sense of De Concini and Procesi (in our case, we need the general construction of De Concini and Springer).

Keywords: Wonderful group completion; Line bundle; Cohomology; Borel–Weil–Bott theorem

0. Introduction

We present here a Borel–Weil–Bott type theorem for the wonderful completion X of a split adjoint semisimple algebraic group G over a field k of characteristic zero. We roughly explain the situations. Precise definitions are found in Section 1. Let $\sigma \in \{+,-\}^r$. Let $\mathcal{L}_\lambda$ be a line bundle on X corresponding to $\lambda \in \operatorname{Pic} X \cong P$ (cf. De Concini and Procesi [6, Section 8]). Let P_σ and Q_σ be subsets of the weight and root lattices of G_{sc} defined by the pairing with $\{\alpha_i^\vee\}$ and $\{\omega_i^\vee\}$, respectively. Let μ_+ be the unique weight in $W \star \mu$ such that $\mu_+ + \rho$ is dominant. Let l_μ be the minimal length of $w \in W$ such that $w \star \mu = \mu_+$. Let n_σ be the number of $-$ in σ. Then, our result is expressed as follows:

$$H^*(X, \mathcal{L}_\lambda) \cong \bigoplus_{\sigma \in \{+,-\}^r} \bigoplus_{\mu \in [\lambda + Q_\sigma] \cap P_\sigma} V(\mu_+) \boxtimes V(-w_0\mu_+)[-2l_\mu - n_\sigma].$$

☆ The author is supported by JSPS Research Fellowship for Young Scientists.
E-mail address: syuchan@ms.u-tokyo.ac.jp.

0021-8693/03/$ – see front matter
PII: S0021-8693(02)00569-0

The proof relies on the Grothendieck–Cousin resolution. Before the application of the general theory, we need to determine the finite-dimensional part of differential of the complex.

Rational cohomologies of complete symmetric spaces are completely described by the works of [2,3,6–8]. However, there seemed to be no work exactly on the Borel–Weil–Bott type theorem. Questions of Brion [4,5] are the nearest ones to ours. But the author does not know any material concerning an explicit result on this topic.[1]

1. Statement of the result

Let G be a semisimple adjoint group of rank r over a field k of characteristic zero. We assume G is k-split. Let G_{sc} be its simply connected cover. Put $\mathfrak{g} = \operatorname{Lie} G$. Let $T \subset B \subset G$ be a maximal torus which is k-split and a Borel subgroup of G. Let T_{sc} be the pullback of T to G_{sc}. Let $Q \subset P$ be the root and weight lattices of G_{sc}, respectively. Let $\Delta \subset Q$ be the set of roots of G. Let $\Delta^+ \subset \Delta$ be the set of the positive roots of G with respect to B. Denote B^{opp} the opposite Borel group of B with respect to T. Put $U = [B, B]$ and $U^{\mathrm{opp}} = [B^{\mathrm{opp}}, B^{\mathrm{opp}}]$. Let $W = N_G(T)/Z_G(T)$ be the Weyl group of G. Let $l : W \to \mathbf{Z}_{\geqslant 0}$ be the length function of W defined by Δ^+. Let $w_0 \in W$ be the longest element of W. Let $V(\lambda)$ be an irreducible representation of G_{sc} with a highest weight λ. Let X be the wonderful completion of G in the sense of De Concini and Procesi (cf. [9]). We have $\operatorname{Pic} X \cong P$ by [6, Section 8] or [12, Lemma 2.4]. Let p_0 be the unique $B^{\mathrm{opp}} \times B^{\mathrm{opp}}$-fixed point of X. For each $\lambda \in P$, we have a line bundle $\mathcal{L}_\lambda$ of X such that $\mathcal{L}_\lambda \otimes k(p_0)$ has weight $(\lambda, -w_0\lambda)$ as a $(T_{\mathrm{sc}} \times T_{\mathrm{sc}})$-module. Put $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$. For each $w \in W$, we define the star-action $w \star \lambda$ of W on P by $w \star \lambda := w(\lambda + \rho) - \rho$. Let λ_+ be the unique weight in $W \star \lambda$ such that $\lambda_+ + \rho$ is dominant. Put $l_\lambda := \min\{l(w) \mid w \star \lambda = \lambda_+\}$. Let $(\alpha_1, \dots, \alpha_r)$ and $(\alpha_1^\vee, \dots, \alpha_r^\vee)$ be the set of positive simple roots and positive simple coroots of G. For each $\sigma := (\sigma_1, \dots, \sigma_r) \in \{+, -\}^r$, we define $P_\sigma \subset P$ as follows:

$$P_\sigma := \bigcap_{i=1}^{r} \bigl\{\mu \in P \bigm| \alpha_i^\vee(\mu) < 0 \text{ iff } \sigma_i = -\bigr\}.$$

Put $Q_\sigma := \{\mu = \sum_{j=1}^r a_k\alpha_k \in Q \mid a_k > 0 \text{ iff } \sigma_k = -\}$, $n_\sigma = \#\{i \mid \sigma_i = -\}$, and $P_+ := P_{(+,\dots,+)}$.

Theorem 1.1. *For any $\lambda \in P \cong \operatorname{Pic} X$, we have the following description of $H^*(X, \mathcal{L}_\lambda)$:*

$$H^*(X, \mathcal{L}_\lambda) \cong \bigoplus_{\sigma \in \{+,-\}^r} \bigoplus_{\mu \in [\lambda + Q_\sigma] \cap P_\sigma} V(\mu_+) \boxtimes V(-w_0\mu_+)[-2l_\mu - n_\sigma].$$

We understand that the RHS is a sum of complex of vector spaces of degree $2l_\mu + n_\sigma$.

[1] During the reviewing period of this article, I have informed that Alexis Tchoudjem [13] also obtained this result independently. Although his expression looks bit different from that of this article, the assertion is the same. I heard that he is also preparing the full version [14] of [13].

2. Proof of theorem

We have natural inclusion $\iota : \mathcal{O}_0 := G/B^{\mathrm{opp}} \times G/B \hookrightarrow X$. $\mathcal{O}_0$ is the unique closed orbit of X. We have $p_0 = B^{\mathrm{opp}} \times w_0 B \in G/B^{\mathrm{opp}} \times G/B$. $W \times W$ operates on $\mathcal{O}_0^{T\times T}$, the $(T\times T)$-fixed points of $\mathcal{O}_0$, transitively. Note that the set of $(T\times T)$-fixed points of X is $\mathcal{O}_0^{T\times T}$, too. For each $(\lambda_1,\lambda_2) \in P \oplus P$, we denote the formal character of the corresponding one-dimensional representation by $\mathrm{e}^{(\lambda_1,\lambda_2)}$. For each $(T_{\mathrm{sc}} \times T_{\mathrm{sc}})$-module V, we denote its formal character by $\operatorname{ch} V$.

Lemma 2.1. *For each $(w_1, w_2) \in W \times W$, we have*

$$\operatorname{ch} T_{(w_1,w_2)p_0} X = \sum_{i=1}^{r} \mathrm{e}^{(w_1\alpha_i, -w_2w_0\alpha_i)} + \sum_{\alpha\in\Delta^+} \big(\mathrm{e}^{(w_1\alpha,0)} + \mathrm{e}^{(0,-w_2w_0\alpha)}\big).$$

We have $\operatorname{ch}(\mathcal{L}_\lambda \otimes k((w_1,w_2)p_0)) = \mathrm{e}^{(w_1\lambda, -w_2w_0\lambda)}$.

Proof. For $w_1 = w_2 = \mathrm{id}$, the weight of natural torus action $(t_1,t_2) : g \mapsto t_1 g t_2^{-1}$ of the tangent bundle $T_{p_0}\mathcal{O}_0$ is given by the second term of the above description. The first term follows from [6, Corollary 8.2] and adjunction. $\operatorname{Pic} X$ is a rational linear combination of boundary divisors in this case. The other cases are immediately obtained by the $(W\times W)$-action of $P \oplus P$. □

Let $p \in X^{T\times T}$. Consider a one-parameter subgroup $\xi : \mathbb{G}_m \to T \times T$ such that $\xi(\alpha, 0) > 0$ and $\xi(\beta,\alpha) > 0$ for every $\alpha \in \Delta^+$ and $\beta \in \Delta \cup \{0\}$. We have $X^{T\times T} = X^{\operatorname{Im}\xi}$. For each $p \in X^{T\times T}$, we put $Z(p) := \{x \in X \mid \lim_{t\to 0} \xi(t)x = p\}$ (attracting set at $p \in X$). By Bialynicki-Birula's theorem [1, Section 4], each cell $Z(p)$ is affine and gives a paving of X if k is algebraically closed.

Theorem 2.1 ([12, 2.1 and 2.2], [9, 3.10(d)]). *Put $p_1 := (\mathrm{id}, w_0)p_0$. There exists a partial completion $\kappa : \mathbf{A}^r \subset X$ of $T \subset G \subset X$ such that $[\mathbf{A}^r]^{T\times T} = p_1$. We have an affine open embedding $N(p_1) := U \times \mathbf{A}^r \times U^{\mathrm{opp}} \hookrightarrow X$ defined by κ and multiplication.*

Remark 2.1. This construction seems to be essentially identical to that appeared in Shinobu Sato's thesis [11]. The author wants to express his thanks to Prof. Toshio Oshima to inform me of that article.

Let $(w_1,w_2) \in W\times W$. Let $(\widetilde{w}_1, \widetilde{w}_2) \in N_{G\times G}(T\times T)$ be an arbitrary representative. By twisting $N(p_1)$ by $(\widetilde{w}_1,\widetilde{w}_2)$, we have an affine open neighborhood $(\widetilde{w}_1,\widetilde{w}_2)p_1 \in \widetilde{w}_1 N(p_1)\widetilde{w}_2^{-1}$. $N(p_1)$ is a geometric representation of $T\times T$ in the sense of Kempf [10]. In particular, $\widetilde{w}_1 N(p_1)\widetilde{w}_2^{-1}$ only depends on the choice of (w_1,w_2). We denote $\widetilde{w}_1 N(p_1)\widetilde{w}_2^{-1}$ by $N(p)$ for $p = (w_1,w_2)p_1$. We have $Z(p) \subset N(p)$.

For each simple root α_i, there exists a $(G\times G)$-stable subvariety H_i of codimension one such that $(\alpha_i, -w_0\alpha_i) = [H_i] \in P \cong \operatorname{Pic} X$. For each $J \subset [1,r]$, we put $Z(p)_J :=$

$Z(p) \cap \bigcap_{j \in J} H_j$. By the above arguments, we obtain that every cell $Z(p)$ is defined over k. Put

$$Z_n := \bigcup_{\substack{p \in X^{T \times T},\ J \subset [1,r] \\ \operatorname{codim} Z(p)_J \geqslant n}} Z(p)_J.$$

We use the Grothendieck–Cousin complex associated to the decreasing stratification $\{Z_n\}_{n \in \mathbf{Z}}$. $\{Z_n\}_{n \in \mathbf{Z}}$ is well-defined for arbitrary k. Each $Z(p)_J$ is a closure of a $(B \times B)$-orbit in $N(p)$. Z_n is closed for all $n \in \mathbf{Z}$. $Z(p)$ is defined by zeros of $(T \times T)$-characters in $N(p)$. We can apply Kempf's arguments. We denote the set of weights appearing in $T_p N(p)$ by $\mathrm{wt}(p)$.

Theorem 2.2 (Kempf [10, 11.10]). *Let $p \in X^{T \times T}$, $\operatorname{codim} Z(p) = i$, and λ_p be the character of $(T_{\mathrm{sc}} \times T_{\mathrm{sc}})$-module $\mathcal{L}_\lambda \otimes k(p)$. We have*

$$\operatorname{ch} H^i_{Z(p)}\bigl(N(p), \mathcal{L}_\lambda\bigr) = \mathrm{e}^{\lambda_p} \prod_{\substack{\alpha \in \mathrm{wt}(p) \\ \xi(\alpha) > 0}} \frac{1}{1 - \mathrm{e}^{-\alpha}} \prod_{\substack{\alpha \in \mathrm{wt}(p) \\ \xi(\alpha) < 0}} \frac{\mathrm{e}^{\alpha}}{1 - \mathrm{e}^{\alpha}}.$$

We denote the above formal sum by $h(\lambda, p)$.

We define $d := (\mathrm{id}, -w_0 \star) : P \to P \oplus P$.

Lemma 2.2. *Let $(w_1, w_2) \in W \times W$. The following three conditions are equivalent.*

(1) $w_1 = w_0 w_2 w_0$.
(2) $(w_1, w_2) \star d(P) = d(P)$.
(3) $(w_1, w_2) \star d(P) \cap d(P_+) \neq \emptyset$.

Here $\star$-action should be considered component-wise.

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are trivial. Let $(w_1 \star \lambda, -(w_2 w_0) \star \lambda) \in d(P_+)$. If $w_1 \star \lambda \in P_+$, then we have $-(w_2 w_0) \star \lambda \in P_+$ only if $w_1 = w_0 w_2 w_0$. We have (3) $\Rightarrow$ (1). $\square$

We denote $w_0 w w_0$ by $\overline{w}$. Put $\mathfrak{b} = \operatorname{Lie} B$.

Definition 2.1. We define the full subcategory $\mathcal{O}$ of the category of $(\mathfrak{g} \times \mathfrak{g})$-modules which consists of object V with the following properties:

(1) V has a $T_{\mathrm{sc}} \times T_{\mathrm{sc}}$-isotypical decomposition $V = \bigoplus_{\gamma \in P \oplus P} V_\gamma$ such that $\dim V_\gamma < \infty$.
(2) For each $v \in V_\gamma$, we have $\dim(U(\mathfrak{b}) \oplus U(\mathfrak{b}))v < \infty$.

(3) There exists some $\gamma \in P \oplus P$ such that

$$V_\mu \neq \{0\} \quad \Rightarrow \quad \mu \in \gamma - \sum_{i=1}^{r} \mathbf{Z}_{\geqslant 0}(\alpha_i, 0) - \sum_{\substack{\alpha \in \Delta^+ \\ \beta \in \Delta \cup \{0\}}} \mathbf{Z}_{\geqslant 0}(\beta, \alpha).$$

Let $K(\mathcal{O})$ be the Grothendieck group of $\mathcal{O}$. Denote the element in $K(\mathcal{O})$ corresponding to $M \in \mathcal{O}$ by $[M]$. Let $\mathcal{O}^f$ (respectively $\mathcal{O}^\infty$) be the full subcategory of $\mathcal{O}$ consisting of modules such that every irreducible constituents are finite (respectively infinite) dimensional. Irreducible constituents of an element of $\mathcal{O}$ are completely determined by its character. We have $K(\mathcal{O}) \cong K(\mathcal{O}^f) \oplus K(\mathcal{O}^\infty)$. Let $[M]_f$ be the image of $[M]$ in $K(\mathcal{O}^f)$.

Note that a $(\mathfrak{g} \times \mathfrak{g})$-module with character $h(\lambda, (w_1, w_2)p_0)$ is contained in $\mathcal{O}$ by a character computation.

Corollary 2.1. *A $(\mathfrak{g} \times \mathfrak{g})$-module with character $h(\lambda, (w_1, w_2)p_0)$ has a finite-dimensional irreducible constituent only if $w_1 = \overline{w}_2$.*

Proof. Put

$$\Psi(\lambda) := \left\{ \gamma \in P \;\middle|\; \gamma = \lambda - \sum_{i=1}^{r} n_i \alpha_i n_i \in \mathbf{Z} \text{ and } n_i \geqslant 0 \text{ iff } -w_2 w_0 \alpha_i \in \Delta^+ \right\}.$$

Applying Lemma 2.1 to Theorem 2.2, we have

$$\begin{aligned}
h\bigl(\lambda, (w_1, w_2)p_0\bigr) &= \mathrm{e}^{(w_1\lambda, -w_2 w_0 \lambda)} \prod_{\substack{\alpha \in \mathrm{wt}(p) \\ \xi(\alpha) > 0}} \frac{1}{1 - \mathrm{e}^{-\alpha}} \prod_{\substack{\alpha \in \mathrm{wt}(p) \\ \xi(\alpha) < 0}} \frac{\mathrm{e}^{\alpha}}{1 - \mathrm{e}^{\alpha}} \\
&= \mathrm{e}^{(w_1\lambda, -w_2 w_0 \lambda)} \operatorname{ch}\bigl(M(w_1 \star 0) \boxtimes M(w_2 \star 0)\bigr) \\
&\quad \times \prod_{-w_2 w_0 \alpha_i \in \Delta^+} \frac{1}{1 - \mathrm{e}^{-(w_1 \alpha_i, -w_2 w_0 \alpha_i)}} \\
&\quad \times \prod_{-w_2 w_0 \alpha_i \in -\Delta^+} \frac{\mathrm{e}^{(w_1 \alpha_i, -w_2 w_0 \alpha_i)}}{1 - \mathrm{e}^{(w_1 \alpha_i, -w_2 w_0 \alpha_i)}} \\
&= \sum_{\gamma \in \Psi(\lambda)} \operatorname{ch}\bigl(M(w_1 \star \gamma) \boxtimes M\bigl(-(w_2 w_0) \star \gamma\bigr)\bigr).
\end{aligned}$$

We have $[H^i_{Z(p)}(N(p), \mathcal{L}_\lambda)] = \sum_{\gamma \in \Psi(\lambda)} [M(w_1 \star \gamma) \boxtimes M(-(w_2 w_0) \star \gamma)]$ in $K(\mathcal{O})$. Here $M(\lambda)$ is the Verma module of highest weight λ. $M(\lambda)$ has a finite-dimensional irreducible constituent only if $\lambda \in P_+$. Applying Lemma 2.2, we obtain the result. $\square$

Lemma 2.3. *Let $w \in W$. $w \star P_+ \subset P_\sigma$ if and only if $l(s_{\alpha_i} w) > l(w)$ for $\sigma_i = +$ and $l(s_{\alpha_i} w) < l(w)$ for $\sigma_i = -$.*

Proof. Let $l(s_{\alpha_i} w) > l(w)$. We have $l(w^{-1}s_{\alpha_i}^{-1}) > l(w^{-1})$. $-w^{-1}\alpha_i^\vee$ is a negative coroot, since $w^{-1}s_{\alpha_i}$ first brings $\alpha_i^\vee$ to negative coroots and no positive coroot goes through negative coroot and back to positive coroots under the reduced expression of w^{-1}. If λ is a strictly dominant weight, we have $\alpha_i^\vee(w\lambda) = w^{-1}(\alpha_i^\vee)(\lambda) > 0$. The above argument also produces the reverse implication. Moreover, we have $l(s_{\alpha_i} w) < l(w) \Rightarrow w(\alpha_i^\vee)(\lambda) < 0$ by the same argument. Collecting these inequalities for all $i = 1, \dots, r$, we obtain the result. □

Lemma 2.4. *Let $w \in W$ and choose $\tau \in \{+,-\}^r$ such that $w^{-1} \star P_+ \subset P_\tau$. Let denote $H(\lambda, (w, \overline{w})p_0)$ a $(\mathfrak{g} \times \mathfrak{g})$-module with character $h(\lambda, (w, \overline{w})p_0)$. Then, we have*

$$\bigl[H\bigl(\lambda, (w, \overline{w})p_0\bigr)\bigr]_f = \sum_{\mu \in [\lambda + Q_\tau] \cap w^{-1} \star P_+} \bigl[V(\mu_+) \boxtimes V(-w_0\mu_+)\bigr].$$

Proof. We have only to determine

$$w \star \left\{ \lambda + \sum_{i=1}^{r} n_i \alpha_i \;\middle|\; n_i \leqslant 0 \text{ iff } w\alpha_i \in \Delta^+ \right\} \cap P_+.$$

We have $w\alpha_i \in \Delta^+$ iff $l(ws_{\alpha_i}) > l(w) \Leftrightarrow l(s_{\alpha_i} w^{-1}) > l(w^{-1})$. Put $\sigma^w \in \{+,-\}^r$ such that $\sigma_i^w = +$ iff $w\alpha_i \in \Delta^+$. By Lemma 2.3, we have $w^{-1} \star P_+ \subset P_{\sigma^w}$. Applying $w^{-1}\star$ to the above set and setting $\tau = \sigma^w$, we obtain the result. □

We may use the notation σ^w introduced in the proof of above lemma hereafter.

Corollary 2.2. *We have* $\operatorname{codim} Z((w, \overline{w})p_0) = 2l(w) + n_{\sigma^w}$.

Proof. We count negative eigenvectors (= destabilizing directions) in the character formula of Lemma 2.1 with respect to the $\mathbb{G}_m$-action via ξ. For the terms $\{(w\alpha, 0)\}_{\alpha \in \Delta^+}$, we have $l(w)$ negative eigenvectors. For the terms $\{(0, -ww_0\alpha)\}_{\alpha \in \Delta^+}$, we have $l(\overline{w}) = l(w)$ negative eigenvectors. For the terms $\{(w\alpha_i, -ww_0\alpha_i)\}_{i=1}^r$, we have n_{σ^w} negative eigenvectors. The number of negative eigenvectors are $2l(w) + n_{\sigma^w}$. □

Proposition 2.1. *Let $u_1, u_2, w_1, w_2 \in W$. If $Z((u_2, \overline{w}_2)p_0)_J \subset \overline{Z((u_1, \overline{w}_1)p_0)_{J'}}$, then we have $w_2 \geqslant w_1$. Here $\geqslant$ means the Bruhat order.*

Proof. Put $p = (u_1, \overline{w}_1)p_0$ and $q = (u_2, \overline{w}_2)p_0$. For each $w \in W$, we put $I_w := \{i \in \{1, \dots, r\} \mid w\alpha_i \in \Delta^+\}$. We have $\{\alpha_i\}_{i \in I_w} \subset w^{-1}\Delta^+ \cap \Delta^+$. Let $B \subset P_{I_w}$ be a parabolic subgroup of G corresponding to $\{\alpha_i\}_{i \in I_w}$. Put L_{I_w}(respectively U_{I_w}) be the Levi (respectively unipotent) component of the Levi decomposition of P_{I_w}. Put $M_{I_w} := [L_{I_w}, L_{I_w}]$. Put $C_w := \bigcap_{i \notin I_w} \ker(\alpha_i) \subset \mathbf{A}^r$. By the description of Theorem 2.1 and the arguments after that, we have

$$Z(p) \cong \bigl(\tilde{u}_1 U \tilde{u}_1^{-1} \cap U\bigr)\tilde{u}_1 C_{w_1} \widetilde{w}_1^{-1}\bigl(\widetilde{w}_1 U^{\mathrm{opp}} \widetilde{w}_1^{-1} \cap U^{\mathrm{opp}}\bigr)\widetilde{w}_0 \subset N(p).$$

We have $Z(p) \subset \bigcap_{i \notin I_{w_1}} H_i$. There is a natural fibration $\pi_{I_{w_1}} : \bigcap_{i \notin I_{w_1}} H_i \to G/P^{\mathrm{opp}}_{I_{w_1}} \times G/P_{I_{w_1}}$ by [6, 5.2]. We denote $M_{I_{w_1}}/Z(M_{I_{w_1}})$ by $M'_{I_{w_1}}$. The fiber of $\pi_{I_{w_1}}$ is isomorphic to the wonderful completion of $M'_{I_{w_1}}$. Put $U_{w_1} := \widetilde{w}_1 U \widetilde{w}_1^{-1} \cap U$ and $U^{\mathrm{opp}}_{w_1} := \widetilde{w}_1 U^{\mathrm{opp}} \widetilde{w}_1^{-1} \cap U^{\mathrm{opp}}$. We put

$$\check{Z}(p) := \left(U_{u_1} \cap \tilde{u}_1 U_{I_{w_1}} \tilde{u}_1^{-1}\right) \tilde{u}_1 M'_{I_{w_1}} \widetilde{w}_1^{-1} \left(U^{\mathrm{opp}}_{w_1} \cap \widetilde{w}_1 U^{\mathrm{opp}}_{I_{w_1}} \widetilde{w}_1^{-1}\right) \widetilde{w}_0.$$

We have an embedding $\overline{\check{Z}(p)} \supset \overline{Z(p)}$. Put $p' = (\mathrm{id}, \overline{w}_1) p_0$. Then $\overline{\check{Z}(p')} = \overline{Z(p')}$. Since $\pi_{I_{w_1}}$ is a $(B \times B)$-equivariant fibration, we obtain

$$\pi_{I_{w_1}}\left(\check{Z}(p)\right) = \left(U_{u_1} \cap \tilde{u}_1 U_{I_{w_1}} \tilde{u}_1^{-1}\right) \pi_{I_{w_1}}(p) \left(U_{\overline{w}_1} \cap \widetilde{\overline{w}}_1 U_{I_{\overline{w}_1}} \widetilde{\overline{w}}_1^{-1}\right).$$

Let $W_{I_{w_1}}$ be a subgroup of W generated by simple reflections associated to $\{\alpha_i\}_{i \in I_{w_1}}$. The Bruhat cells ($B^{\mathrm{opp}}$-orbits) on $G/P_{I_{w_1}}$ are parametrized by $W/W_{I_{w_1}}$, where w_1 is the shortest element of $w_1 W_{I_{w_1}}$. Consider a projection $\mathrm{pr} : \mathcal{O}_0 \overset{\iota}{\hookrightarrow} \bigcap_{i \notin I_{w_1}} H_i \xrightarrow{\pi_{I_{w_1}}} G/P^{\mathrm{opp}}_{I_{w_1}} \times G/P_{I_{w_1}}$. Since this is a fibration, we have

$$Z(q) \cap \mathcal{O}_0 \subset \mathrm{pr}^{-1}\left(\overline{\pi_{I_{w_1}}(Z(q)_J)}\right) \subset \mathrm{pr}^{-1}\left(\overline{\pi_{I_{w_1}}(Z(p)_{J'})}\right) \subset \overline{Z(p') \cap \mathcal{O}_0}.$$

Closure relation on $\mathcal{O}_0$ is determined by Bruhat order. Therefore, $Z(q)_J \subset \overline{Z(p)_{J'}}$ only if $w_2 \geqslant w_1$ by Bruhat order. □

Corollary 2.3. *Under the assumptions of Proposition* 2.1, *assume further* $w_1 = w_2$. *Then we have* $u_2 > u_1$.

Proof. We use the same notations as in the proof of Proposition 2.1. We put $w = w_1$ $(= w_2)$. We prove the assertion case by case. If $u_1 \in u_2 W_{I_w}$, then we have $\pi_{I_w}(Z(p)) = \pi_{I_w}(Z(q))$. In particular, $Z(p) \subset \check{Z}(p)$ is a dense inclusion iff u_1 is the shortest element of $u_2 W_{I_w}$. Thus, the assertion follows from the Bruhat decomposition of a fiber M'_{I_w} of $\check{Z}(q) \to \pi_{I_w}(Z(q))$ in this case. If $u_1 \notin u_2 W_{I_w}$, we have a closure relation of the orbits on $G/P^{\mathrm{opp}}_{I_w} \times G/P_{I_w}$ by taking projection π_{I_w}. By the assumption $w_1 = w_2$, the closure relation requires $u_2 > u_1$ in this case. Thus, we have $u_2 > u_1$ in every case. □

Corollary 2.4. *Under the assumptions of Proposition* 2.1, *we have* $\dim Z((w_1, \overline{w}_1) p_0)_{J'} \geqslant \dim Z((w_2, \overline{w}_2) p_0)_J + 2$ *if* $w_1 \neq w_2$.

Proof. We have $l(w_1) = l(\overline{w}_1)$. Each Bruhat cell of type $Z((w_1, \overline{w}_1) p_0) \cap \mathcal{O}_0 = Z((w_1, \overline{w}_1) p_0)_J \cap \mathcal{O}_0$ is even-dimensional. By Theorem 2.1 and the arguments after it, we also have the $\#\{i \in J' \mid w_1^{-1} \alpha_i \in \Delta^+\} \geqslant \#\{i \in J \mid w_2^{-1} \alpha_i \in \Delta^+\}$. By Proposition 2.1, dimensions must differ by at least two. □

We put $Z(p)_i := Z(p) \cap Z_i$. We have the following natural isomorphism by [10, 12.6 and 12.7]:

$$H^i_{Z_i/Z_{i+1}}(X, \mathcal{L}_\lambda) \cong \bigoplus_{p \in \mathcal{O}^{T \times T}} H^i_{Z(p)_i \setminus Z(p)_{i+1}}\bigl(N(p) \setminus Z(p)_{i+1}, \mathcal{L}_\lambda\bigr).$$

By [10, Lemma 7.7, 7.9(excision) and Theorem 12.5], we consider it as an isomorphism of $(\mathfrak{g} \times \mathfrak{g})$-modules. We have

$$H^i_{Z(p)_i \setminus Z(p)_{i+1}}\bigl(N(p) \setminus Z(p)_{i+1}, \mathcal{L}_\lambda\bigr) \cong H^i_{Z(p)_i/Z(p)_{i+1}}\bigl(N(p), \mathcal{L}_\lambda\bigr). \tag{1}$$

Let denote $\psi_{p,i} : H^i_{Z_i/Z_{i+1}}(X, \mathcal{L}_\lambda) \to H^i_{Z(p)_i}(N(p), \mathcal{L}_\lambda)$ the natural projection, $\phi_{p,i} : H^i_{Z(p)_i}(N(p), \mathcal{L}_\lambda) \to H^i_{Z_i/Z_{i+1}}(X, \mathcal{L}_\lambda)$ the natural inclusion, and $d_i : H^i_{Z_i/Z_{i+1}}(X, \mathcal{L}_\lambda) \to H^{i+1}_{Z_{i+1}/Z_{i+2}}(X, \mathcal{L}_\lambda)$ the differential of the Grothendieck–Cousin complex. For two factors $Z(p)_i$ and $Z(q)_{i+1}$, we can construct an induced differential $d_i^{p,q} = \psi_{q,i+1} \circ d_i \circ \phi_{p,i}$; $d_i^{p,q} : H^i_{Z(p)_i}(N(p), \mathcal{L}_\lambda) \to H^{i+1}_{Z(q)_{i+1}}(N(q), \mathcal{L}_\lambda)$ is nonzero only if $\mathcal{Z} \subset \overline{Z(p)_i}$ for some irreducible constituent $\mathcal{Z} \subset Z(q)_{i+1}$ (cf. [10, Remark after Lemma 12.6]). Let $i = \operatorname{codim} Z(p)$. By (1) we can consider the following subcomplex of the Grothendieck–Cousin complex $\mathcal{C}^\bullet(p)$:

$$0 \to H^i_{Z(p)_i/Z(p)_{i+1}}\bigl(N(p), \mathcal{L}_\lambda\bigr) \to H^{i+1}_{Z(p)_{i+1}/Z(p)_{i+2}}\bigl(N(p), \mathcal{L}_\lambda\bigr) \to \cdots.$$

We have $H^\bullet(\mathcal{C}^\bullet(p)) \cong H^\bullet_{Z(p)}(N(p), \mathcal{L}_\lambda)$ (we considered the above as a Grothendieck–Cousin complex of $\mathcal{H}^i_{Z(p)}(\mathcal{L}_\lambda)$ on $N(p)$). Collecting these morphisms (and other morphisms), we have the following spectral sequence:

$$E_0^{i,j} := \bigoplus_{\operatorname{codim} Z(p) = i} H^{i+j}_{Z(p)_{i+j}/Z(p)_{i+j+1}}\bigl(N(p), \mathcal{L}_\lambda\bigr) \Rightarrow H^{i+j}(X, \mathcal{L}_\lambda).$$

We put $E^{i,j}(p) := H^{i+j}_{Z(p)_{i+j}/Z(p)_{i+j+1}}(N(p), \mathcal{L}_\lambda)$. We have $H^l(E^{i,\bullet}(p)) = 0$ for $l > 0$. By Proposition 2.1 and Corollary 2.3, if $d^{p,q}_{i+j} : E^{i,j}(p) \to E^{i,j+1}(q)$ is nonzero and $q \neq p$, then we have $d^{q,p}_{i+j} = 0$. Thus, the above spectral sequence satisfies $E_1^{i,j} = 0$ for $j > 0$. In particular, $E_*^{i,j}$ is E_2-degenerate. We have a direct decomposition of E_1-term differential $d_i^1 := \bigoplus D_i^{p,q}$ by using $D_i^{p,q} : H^0(E^{i,\bullet}(p)) \to H^0(E^{i+1,\bullet}(q))$. By Corollary 2.4, we have $D_i^{(w_1,\overline{w}_1)p_0,(w_2,\overline{w}_2)p_0} = 0$ for all $w_1, w_2 \in W$ and $i \in \mathbf{Z}$. Hence, we have

$$\sum_{w \in W} \bigl[H^0\bigl(E^{i,\bullet}\bigl((w, \overline{w})p_0\bigr)\bigr)\bigr]_f = \sum_{p \in X^{T \times T}} \bigl[H^{\operatorname{codim} Z(p)}_{Z(p)}\bigl(N(p), \mathcal{L}_\lambda\bigr)\bigr]_f = \bigl[H^*(X, \mathcal{L}_\lambda)\bigr]_f.$$

The term of Lemma 2.4 is of the degree $2l(w) + n_{\sigma^w}$ by Corollary 2.2. Every sheaf cohomology of a line bundle must be finite dimensional. Therefore, we obtain the result.

Acknowledgments

The author expresses his sincere gratitude to his research supervisor Prof. H. Matumoto. The author expresses his thanks to Prof. T.A. Springer for suggesting to use the Grothendieck–Cousin complex. The author expresses his gratitude to A. Tchoudjem for pointing out an error in pre-final version and giving to us a reprint of his article [13] in the Conference held at CRM at May 2002. The author also expresses his thanks to Professors M. Brion, J.C. Jantzen, T. Katsura, F. Knop, T. Oshima, T. Oda, T. Uzawa, Dr. H. Kawabi, Y. Terashima, and A. Nukada. Finally, it is his pleasure to express his gratitude to the anonymous referee for his tolerance, careful reading, and helpful remarks.

References

[1] A. Bialynicki-Birula, Some theorems on actions of algebraic groups, Ann. of Math. 98 (1973) 480–497.
[2] E. Bifet, On complete symmetric varieties, Adv. Math. 80 (2) (1990) 225–249.
[3] E. Bifet, C. De Concini, C. Procesi, Cohomology of regular embeddings, Adv. Math. 82 (1990) 1–34.
[4] M. Brion, Une extension du théorème de Borel–Weil, Math. Ann. 286 (1990) 655–660.
[5] M. Brion, Représentations des groupes réductifs dans des espaces de cohomologie, Math. Ann. 300 (1994) 589–604; Corrigendum: Math. Ann. 301 (1995) 821–822.
[6] C. De Concini, C. Procesi, Complete symmetric varieties, in: Invariant Theory, Montecatini, 1982, in: Lecture Notes in Math., Vol. 996, Springer, Berlin, 1983, pp. 1–44.
[7] C. De Concini, C. Procesi, Cohomology of compactifications of algebraic groups, Duke Math. J. 53 (3) (1986) 585–594.
[8] C. De Concini, T.A. Springer, Betti numbers of complete symmetric varieties, in: Geometry Today, Rome, 1984, in: Progr. Math., Vol. 60, Birkhäuser Boston, 1985, pp. 87–107.
[9] C. De Concini, T.A. Springer, Compactification of symmetric varieties, Transform. Groups 4 (1999) 273–300.
[10] G. Kempf, The Grothendieck–Cousin complex of an induced representation, Adv. Math. 29 (1978) 310–396.
[11] Shinobu Sato, A completion of $G = \mathrm{Int}(\mathfrak{g})$ and $\mathbf{C}$-boundary of unipotent variety, Master's thesis, University of Tokyo, 1982 (in Japanese).
[12] E. Strickland, A vanishing theorem for group compactifications, Math. Ann. 277 (1987) 165–171.
[13] A. Tchoudjem, Cohomologie des fibrés en droites sur la compactification magnifique d'un group semi-simple adjoint, C. R. Acad. Sci. Paris 334 (6) (2002) 441–444.
[14] A. Tchoudjem, Représentations d'algèbres de Lie dans des groupes de cohomologie à support, Thesis, Institut Fourier, Université de Grenoble I, 2002.